# NESTED COISOTROPICS AND SECOND MICROLOCALIZATION

Abstract. Our first goal is to understand the relationship between second microlocal pseudodifferential calculi $\Psi_{2,h}(\mathcal{C}_1)$, $\Psi_{2,h}(\mathcal{C}_2)$ associated with nested coisotropic submanifolds $\mathcal{C}_2 \subset \mathcal{C}_1$. Then we consider the relationship between the corresponding second wavefront sets: ${}^2\mathrm{WF}_{\mathcal{C}_1} \subset SN(\mathcal{C}_1)$, ${}^2\mathrm{WF}_{\mathcal{C}_2} \subset SN(\mathcal{C}_2)$.

## 1. Introduction

In this paper, we consider chains of linear coisotropic submanifolds of $T^*\mathbb{T}^n$. By chains, we mean sequences of nested coisotropics

$$\mathcal{C}_p \subset \ldots \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset T^*\mathbb{T}^n.$$

The codimension of $\mathcal{C}_{j+1}$ is strictly greater than that of $\mathcal{C}_j$, so $p \leq n$.

This project is motivated by the paper [1], in which the authors second microlocalize at sequences of nested *primitive submodules* inside $\mathbb{Z}^n$.

## 2. Calculi associated to nested coisotropics

We speculate a relationship between the second microlocal calculi determined by these coisotropics.

**Conjecture 2.1.** *Let $\mathcal{C}_{j+1} \subset \mathcal{C}_j \subset T^*\mathbb{T}^n$. Let*

$$\beta_{\mathcal{C}_{j+1}} : [T^*\mathbb{T}^n; \mathcal{C}_{j+1}] \longrightarrow T^*\mathbb{T}^n$$

*be the blowdown map for $\mathcal{C}_{j+1}$. Choose $B \in \Psi^{0,0}_{2,h}(\mathcal{C}_{j+1})$ satisfying the condition*

$$ {}^2\mathrm{WF}'_0(B) \cap \beta^{-1}_{\mathcal{C}_{j+1}}(\mathcal{C}_j \backslash \mathcal{C}_{j+1}) = \emptyset. \tag{2.1}$$

*Then*

$$B \circ \Psi_{2,h}(\mathcal{C}_j) \subset \Psi_{2,h}(\mathcal{C}_{j+1}).$$

The idea behind this conjecture is as follows. As $\mathcal{C}_j$ is the larger coisotropic, its spherical normal is smaller than that of $\mathcal{C}_{j+1}$ (i.e., is comprised of *fewer* normal directions). Consider a symbol function in the $\mathcal{C}_j$ calculus. This is a function which may be singular at $\mathcal{C}_j \times \{h = 0\}$, but whose singularity is resolved in the blowup. Since the introduction of fewer normal directions is sufficient to resolve this hypothetical singularity, introducing a greater number of normal directions would certainly resolve such a singularity. However, since the $\mathcal{C}_{j+1}$-total symbol space is the blowup of $\mathcal{C}_{j+1} \times \{0\} \subset \mathcal{C}_j \times \{0\}$, we must first apply a cutoff and specifically consider the singularity of the symbol at $\mathcal{C}_{j+1} \times \{0\}$. So we conjecture that the symbol, after application of cutoff, may be regarded as a symbol in the $\mathcal{C}_{j+1}$ calculus. In Figure 1, $\mathcal{C}_2 \subset \mathcal{C}_1$ is at the center of the sphere ($n = 3$). We are cutting off away from the line segments to the left and right of the sphere. In particular:

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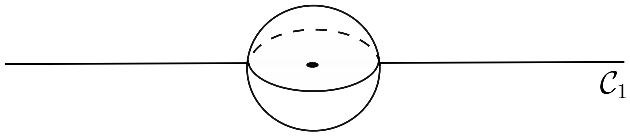


FIGURE 1. Lifting part of $\mathcal{C}_1$ to the $\mathcal{C}_2$-principal symbol space

**Conjecture 2.2.** *Let $\mathcal{C}_{j+1} \subset \mathcal{C}_j$, as above. Let $\Re_{\mathcal{C}_j}$ denote the residual algebra in $\Psi_{2,h}(\mathcal{C}_j)$, and likewise for $\Re_{\mathcal{C}_{j+1}}$. For any operator $B$ fulfilling condition (2.1), we have*

$$B\Re_{\mathcal{C}_j} \subset \Re_{\mathcal{C}_{j+1}}.$$

*More specifically, $B\Re^l_{\mathcal{C}_j} \subset \Re^l_{\mathcal{C}_{j+1}}$ for each $l \in \mathbb{R}$.*

We have proved Conjecture 2.2 in the model case:

**Lemma 2.3.** *For $0 \le p \le (n-1)$ and $q \ge 1$, with $p+q \le n$, let*

$$\mathcal{C}_{j+1} = \mathbb{T}^n \times \{\xi_1 = \ldots = \xi_p = \ldots = \xi_{p+q} = 0\}$$

*and*

$$\mathcal{C}_j = \mathbb{T}^n \times \{\xi_1 = \ldots = \xi_p = 0\}.$$

*Suppose $B \in \Psi^{0,0}_{2,h}(\mathcal{C}_{j+1})$ satisfies the cutoff condition (2.1). Then*

$$B\Re^l_{\mathcal{C}_j} \subset \Re^l_{\mathcal{C}_{j+1}}$$

*for each $l \in \mathbb{R}$.*

Note that since there are fewer characteristic operators for $\mathcal{C}_j$, we have $\Re_{\mathcal{C}_{j+1}} \subset \Re_{\mathcal{C}_j}$. Before proving Lemma 2.3, we give some examples.

**Example 2.4.** This example takes place in $T^*\mathbb{T}^2$. Let $\mathcal{C}_2 = o$ be the zero section, and let $\mathcal{C}_1 = \{\xi_1 = 0\}$. Let $R$ be any element of $\Re^0_{\mathcal{C}_1}$. We will construct an operator $A$ in the $\mathcal{C}_2$-calculus that satisfies condition (2.1), then show that $AR \in \Re^0_{\mathcal{C}_2}$ (i.e., that $AR$ is involutizing w.r.t. $\mathcal{C}_2$). More explicitly, since $hD_{x_1}$, $hD_{x_2}$ generate the module of operators in $\Psi^0_h(\mathbb{T}^2)$ that are characteristic on $\mathcal{C}_2$, we show that

$$h^{-k}(hD_{x_1})^k ARu \in L^2(\mathbb{T}^2) \text{ and } h^{-k}(hD_{x_2})^k ARu \in L^2(\mathbb{T}^2)$$

(for $u \in L^2(\mathbb{T}^2)$ and $k \in \mathbb{Z}_{\ge 0}$).

We want the microsupport of $A$ to be disjoint from the lift of $\mathcal{C}_1 \backslash \mathcal{C}_2$ to $S^{\mathcal{C}_2}_{\mathrm{pr}} = [T^*\mathbb{T}^2; \mathcal{C}_2]$. In this example, condition (2.1) is satisfied if $|\xi_1|$ is greater than $|\xi_2|$ (i.e., $(\xi_1, \xi_2)$ lives in a cone away from $\beta^{-1}_{\mathcal{C}_2}(\mathcal{C}_1 \backslash \mathcal{C}_2)$), **and also** $\xi_1 \ge h$. (More generally, condition (2.1) would hold if $|\xi_1| \ge c|\xi_2|$ for any positive constant $c$, no matter how small.) We therefore define

$$A := {}^h\mathrm{Op}_l\left[\psi\left(\frac{|\xi_1|}{\sqrt{\xi_2^2 + h^2}}\right)\right] \in \Psi^{0,0}_{2,h}(\mathcal{C}_2),$$

where $\psi \in C^\infty(\mathbb{R})$ is supported in $[1, \infty)$. Then, we compute

$$D^k_{x_2} ARu(x) = \int\int \left(\frac{\xi_2}{h}\right)^k e^{\frac{i}{h}(x-y)\cdot\xi} \psi\left(\frac{|\xi_1|}{\sqrt{\xi_2^2 + h^2}}\right) \chi(x,y) Ru(y)\ dy đ\xi$$

$$= \int\int \left(\frac{\xi_2}{h}\right)^k \left(\frac{h}{\xi_1}\right)^k \Delta_{y_1}^{k/2}\left[e^{\frac{i}{h}(x-y)\cdot\xi}\right] \psi\left(\frac{|\xi_1|}{\sqrt{\xi_2^2+h^2}}\right) \chi(x,y)Ru(y)\; dy đ\xi$$

$$= \int\int \left(\frac{\xi_2}{\xi_1}\right)^k e^{\frac{i}{h}(x-y)\cdot\xi}\psi\left(\frac{|\xi_1|}{\sqrt{\xi_2^2+h^2}}\right) \chi(x,y)\Delta_{y_1}^{k/2}Ru(y)\; dy đ\xi + \Psi_{2,h}^{-\infty,0}(\mathcal{C}_1).$$

Recall the convention that $\Delta_{y_1} = -\partial^2/\partial {y_1}^2$. Note that $\Delta_{y_1}^{k/2}Ru \in L^2(\mathbb{T}^2)$ because $R \in \mathfrak{R}^0_{\mathcal{C}_1}$. Note also that $\xi_2/\xi_1 \le 1$ on the microsupport of $A$. This is crucial: if the amplitude becomes any worse, $L^2$-boundedness may fail. Hence, $AR$ is involutizing with respect to $hD_{x_2}$. This argument works even for odd values of $k$, since $h^2\Delta$ taken to a fractional power is well-defined as a pseudodifferential operator.

If instead we apply $D_{x_1}$, we use the fact that

$$D_{x_1}^k ARu = AD_{x_1}^k Ru.$$

Since $R$ is involutizing with respect to $\{\xi_1 = 0\}$, and since the symbol of $A$ belongs to $S^{0,0}\left(S^{\mathcal{C}_2}_{\mathrm{tot}}\right)$, we have $AD_{x_1}^k Ru \in L^2(\mathbb{T}^2)$.

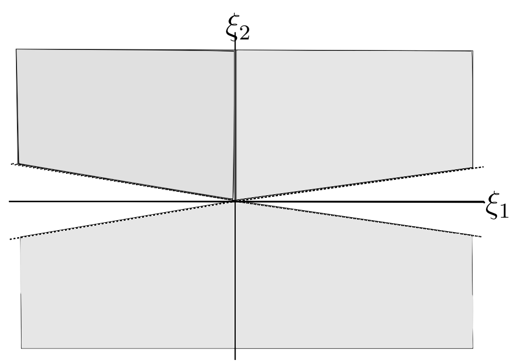


FIGURE 2. Microsupport of cutoff when $\mathcal{C}_2 = o$ and $\mathcal{C}_2 = \{\xi_2 = 0\}$ in $T^*\mathbb{T}^2$

In Figure 2, the shaded region shows the microsupport of the cutoff operator when $\mathcal{C}_2 = o$ and $\mathcal{C}_2 = \{\xi_2 = 0\}$ in $T^*\mathbb{T}^2$ (base variables excluded from figure).

**Example 2.5.** Consider the zero section in $T^*\mathbb{T}^4$, nested inside the codimension two coisotropic $\mathcal{C}_1 = \{\xi_1 = \xi_2 = 0\}$. In this case, $A \in \Psi^{0,0}_{2,h}(\mathbb{T}^4; o)$ satisfying condition (2.1) would be microsupported in $\{\xi_1^2 + \xi_2^2 \ge \xi_3^2 + \xi_4^2\}$. Thus, we would define

$$A := {}^h\mathrm{Op_l}\left[\psi\left(\frac{\sqrt{\xi_1^2+\xi_2^2}}{\sqrt{\xi_3^2+\xi_4^2+h^2}}\right)\right]$$

for $\psi$ as before.

*Proof of Lemma 2.3.* For simplicity, we prove the lemma in the case $l = 0$. Define $A \in \Psi^{0,0}_{2,h}(\mathcal{C}_{j+1})$ as follows:

$$A := {}^h\mathrm{Op_l}\left[\psi\left(\frac{\sqrt{\xi_1^2+\ldots+\xi_p^2}}{\sqrt{c_{p+1}\xi_{p+1}^2+\ldots+c_{p+q}\xi_{p+q}^2+h^2}}\right)\right], \tag{2.2}$$

for $\psi \in C^\infty(\mathbb{R})$ supported on $[1,\infty)$; here, $c_{p+1},\dots,c_{p+q}$ are positive constants. Note that while we used the left quantization to define $A$, any other quantization map would have worked just as well.

Then, for $u \in L^2(\mathbb{T}^n)$ and $R \in \Re^0_{\mathcal{C}_j}$, simultaneously apply the operators $D^{m_{p+1}}_{x_{p+1}},\dots,D^{m_{p+q}}_{x_{p+q}}$ to $ARu(x)$. Set $m = \sum_{i=1}^q m_{p+i}$. Rewrite the phase term as

$$\left[\left(\frac{\xi_1}{h}\right)^2 + \ldots + \left(\frac{\xi_p}{h}\right)^2\right]^{-m/2} \Delta^{m/2}_{y_1\ldots y_p}\left[e^{\frac{i}{h}(x-y)\cdot\xi}\right].$$

Next, integrate by parts, shifting the fractional Laplacian over to the term $Ru(y)$, as the symbol of $A$ is independent of $y$. Finally, use the fact that

$$\frac{\xi_{p+i}^2}{\xi_1^2+\ldots+\xi_p^2} \le \frac{1}{c_{p+i}},\ 1 \le i \le q$$

on the microsupport of $A$, to prove $L^2$-boundedness. Application of $D_{x_1},\dots,D_{x_p}$ is handled using translation invariance, as in Example 2.4.

We include the positive constants in the definition (2.2) so that the conic microsupport of $A$ may be as close to $\beta^{-1}_{\mathcal{C}_{j+1}}(\mathcal{C}_j \backslash \mathcal{C}_{j+1})$ as we like. Therefore, any $B$ satisfying condition (2.1) is microsupported within the elliptic set of one of these operators $A$. Thus, by elliptic regularity, $B$ may be factored as

$$B = A_0 A + S$$

for some $A_0 \in \Psi^{0,0}_{2,h}(\mathcal{C}_{j+1})$ and some $S \in \Re^0_{\mathcal{C}_{j+1}}$. Hence,

$$BR = A_0AR + SR.$$

We just proved that $AR \in \Re^0_{\mathcal{C}_{j+1}}$. Then $A_0AR \in \Re^0_{\mathcal{C}_{j+1}}$, since the residual operators are an ideal.

*Claim:* For $R \in \Re^0_{\mathcal{C}_j}$ and $S \in \Re^0_{\mathcal{C}_{j+1}}$, $S \circ R \in \Re^0_{\mathcal{C}_{j+1}}$.

*Proof of Claim:* We show

$$D^{m_1}_{x_1}\cdots D^{m_{p+q}}_{x_{p+q}} SR : L^2(\mathbb{T}^n) \longrightarrow L^2(\mathbb{T}^n)$$

for any $m_i \in \mathbb{Z}_{\ge 0}$. But this is just the composition of two $L^2$-bounded operators. Certainly $R : L^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$; in fact, we have the much stronger mapping property that $Ru \in I^\infty_{(0)}(\mathcal{C}_j)$ for $u \in L^2(\mathbb{T}^n)$. Finally,

$$D^{m_1}_{x_1}\cdots D^{m_{p+q}}_{x_{p+q}} S : L^2(\mathbb{T}^n) \longrightarrow L^2(\mathbb{T}^n),$$

since $S \in \Re^0_{\mathcal{C}_{j+1}}$. Thus, the claim is proved. Note that the order of the composition is important: it is not the case that $RS \in \Re^0_{\mathcal{C}_{j+1}}$.

Therefore, $BR \in \Re^0_{\mathcal{C}_{j+1}}$. □

Set

$$a(\xi;h) := \psi\left(\frac{\sqrt{\xi_1^2+\ldots+\xi_p^2}}{\sqrt{c_{p+1}\xi_{p+1}^2+\ldots+c_{p+q}\xi_{p+q}^2+h^2}}\right) \in S^{0,0}\left(S^{\mathcal{C}_{j+1}}_{\text{tot}}\right).$$

*Proof of Conjecture 2.1 in model case.* Let $\mathcal{C}_{j+1} \subset \mathcal{C}_j \subset T^*\mathbb{T}^n$ be as in the proof of Lemma 2.3. We ask: Given $A \in \Psi^{0,0}_{2,h}(\mathcal{C}_{j+1})$ of the form (2.2) and $P \in \Psi^{m,l}_{2,h}(\mathcal{C}_j)$, is $A \circ P \in \Psi^{m,l}_{2,h}(\mathcal{C}_{j+1})$? Locally, let $p \in S^{0,0}\left(S^{\mathcal{C}_j}_{\text{tot}}\right)$ be a total symbol for $P$.

First, we show formally that $AP = {}^h\mathrm{Op}_\mathrm{r}(a \cdot p) + AR$ for $R \in \mathfrak{R}^0_{\mathcal{C}_j}$. Due to the reduction theorem, modulo an element of $R \in \mathfrak{R}^0_{\mathcal{C}_j}$, we may choose $P = {}^h\mathrm{Op}_\mathrm{r}(p)$. Therefore, modulo $AR \in \mathfrak{R}^0_{\mathcal{C}_{j+1}}$, we consider ${}^h\mathrm{Op}_\mathrm{l}(a) \circ {}^h\mathrm{Op}_\mathrm{r}(p)$. As in the proof of composition in [4], we compute

$$\begin{aligned}
{}^h\mathrm{Op}_\mathrm{l}(a) \circ {}^h\mathrm{Op}_\mathrm{r}(p) &= (2\pi h)^{-2n} \int e^{\frac{i}{h}(x-z)\cdot\xi} e^{\frac{i}{h}(z-y)\cdot\eta} \chi(x,z)\chi(z,y) a(\xi;h) p(y,\eta;h)\ dz d\xi d\eta \\
&= (2\pi h)^{-n} \int e^{\frac{i}{h}(x-y)\cdot\eta} \chi(x,y) a(\eta;h) p(y,\eta;h)\ d\eta + \Psi^{-\infty,0}_{2,h}(\mathcal{C}_j) \\
&= {}^h\mathrm{Op}_\mathrm{r}(a \cdot p),
\end{aligned}$$

using stationary phase. More generally, for $P \in \Psi^{m,l}_{2,h}(\mathcal{C}_j)$, we have $AP \in \Psi^{m,l}_{2,h}(\mathcal{C}_{j+1}) + \mathfrak{R}^l_{\mathcal{C}_{j+1}}$.

Finally, we have $a \cdot p \in S^{0,0}\left(S^{\mathcal{C}_{j+1}}_{\text{tot}}\right)$. □

In particular, if a Lagrangian $\mathcal{L} \subset \mathcal{C}$, then for each $m, l$

$$B \circ \Psi^{m,l}_{2,h}(\mathcal{C}) \subset \Psi^{m,l}_{2,h}(\mathcal{L}),$$

where $B \in \Psi^{0,0}_{2,h}(\mathcal{L})$ is microsupported away from $\beta_{\mathcal{L}}^{-1}(\mathcal{C} \backslash \mathcal{L})$.

## 3. Second wavefronts of nested coisotropics

Now suppose we have coisotropic submanifolds $\mathcal{C}_2 \subset \mathcal{C}_1 \subset T^*\mathbb{T}^n$. Then

$$\mathcal{C}_2 \times \{h=0\} \subset \mathcal{C}_1 \times \{h=0\} \subset T^*X \times \{h=0\} = \partial(T^*X \times [0,1)) \subset T^*X \times [0,1).$$

Take $p \in \mathcal{C}_2$. Then $T_p(\mathcal{C}_2) \hookrightarrow T_p(\mathcal{C}_1)$. This descends to a map of normal spaces

$$\frac{T_p(T^*X)}{T_p(\mathcal{C}_2)} = N_p(\mathcal{C}_2) \longrightarrow N_p(\mathcal{C}_1) = \frac{T_p(T^*X)}{T_p(\mathcal{C}_1)}.$$

Thus, we have a natural bundle morphism $N(\mathcal{C}_2) \to N(\mathcal{C}_1)$. Hence, there is a canonical map

$$\pi : SN(\mathcal{C}_2) \longrightarrow SN(\mathcal{C}_1).$$

In words, the intuitive idea here is that since $\mathcal{C}_2$ is smaller than $\mathcal{C}_1$, its spherical normal is larger; i.e., there are more (unit) normal directions for $\mathcal{C}_2$ than for $\mathcal{C}_1$. This map $\pi$ "condenses" or "collapses" all the normal directions in $SN(\mathcal{C}_2)$ down to the relatively few normal directions in $SN(\mathcal{C}_1)$. **How?**

We are ready to pose a conjecture relating the second wavefront sets associated to $\mathcal{C}_1$ and $\mathcal{C}_2$. Note that for $l \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$, $\mathcal{C}_2 \subset \mathcal{C}_1$ implies $I^k_{(l)}(\mathcal{C}_2) \subset I^k_{(l)}(\mathcal{C}_1)$, as $\mathfrak{M}_{\mathcal{C}_2} \supset \mathfrak{M}_{\mathcal{C}_1}$.

**Conjecture 3.1.** *Let $l, m \in \mathbb{R}$, $k \in \mathbb{Z}_{\geq 0}$, and $u \in I^k_{(l)}(\mathcal{C}_2)$. Let $S \subset SN(\mathcal{C}_2)$. Then*

$${}^2\mathrm{WF}^{m,l}_{\mathcal{C}_2}(u) \cap S = \emptyset \implies {}^2\mathrm{WF}^{m,l}_{\mathcal{C}_1}(u) \cap \pi(S) = \emptyset.$$

We give the heuristic idea behind this conjecture. Since $\mathcal{C}_2$ is contained in $\mathcal{C}_1$, there are a greater number of characteristic operators associated to the smaller coisotropic $\mathcal{C}_2$. Therefore, all else being equal, it is a stronger condition for a distribution $u$ to have coisotropic regularity with respect to $\mathcal{C}_2$. Translating this into the second microlocal language, it is *easier* for $u$ to have $\mathcal{C}_2$-second wavefront (in $SN(\mathcal{C}_2)$) than $\mathcal{C}_1$-second wavefront (in $SN(\mathcal{C}_1)$). Hence, if there is no $\mathcal{C}_2$-wavefront in some subset $S$ of $SN(\mathcal{C}_2)$, we hypothesize that there is no $\mathcal{C}_1$-wavefront in the corresponding subset $\pi(S)$ of $SN(\mathcal{C}_1)$.

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